PMM U.S.S.R.,Vol.44,pp.681-683 0021-8928/81/5 0681$7.50/0
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UDC 539.3

ON IMPROVING THE CONVERGENCE OF THE HOMOGENEOUS SOLUTION METHOD*

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The problem of the stress state of a short circular cylinder whose side surface is rigidly clamped and its plane end-faces subjected to uniform pressure is considered /1/.

Solution of the three-dimensional problem of a body with mixed conditions at its surfaces is, as a rule, associated with serious difficulties, the most important of which is the slow convergence of computational algorithms.

We introduce the axially symmetric dimensionless cylindrical coordinate system

$$r=\frac{\sqrt{x^2+y^2}}{r_0},\quad \zeta=\frac{z}{h_0}\quad \left(\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{h^2}\frac{\partial^2}{\partial\zeta^2},\quad h=\frac{h_0}{r_0}\right)$$

and formulate the input boundary value problem as follows

$$\nu_1\,\mathrm{grad\,div}\,\mathbf{u}+\Delta\mathbf{u}=0,\quad (\nu_1=(1-2\nu)^{-1}),\quad \mathbf{T}=2\mu\left(\frac{\nu_1-1}{2}\,\mathrm{div}\,\mathbf{u}+\mathrm{def}\,\mathbf{u}\right) \tag{1}$$

$$\mathbf{n}_\zeta\cdot\mathbf{T}=-\mathbf{n}_\zeta\quad (0\leqslant r\leqslant 1),\quad |\zeta|=1,\quad n_\zeta=(0,\,\mathrm{sign}\,\zeta)$$

separating for convenience the condition of the side surface rigid clamping

$$\mathbf{u}=0\quad (r=1,\ |\zeta|\leqslant 1) \tag{2}$$

where r_0 is the radius, $2h_0$ is the cylinder height, ν is the Poisson's coefficient, μ is the shear modulus, $\mathbf{u}$ is the vector of displacements, and $\mathbf{T}$ is the stress tensor.

The Lur'e method of homogeneous solutions /2,3/ associates the differential system (1)—(2) with the problem of determination of the unknown coefficients in formulas

$$2\mu u_r=\left(\frac{\nu}{1+\nu}+\frac{a}{2\nu}\right)r+\mathrm{Re}\sum_{p=1}^{\infty}D_p\left[\left(\frac{\sin\gamma_p}{\gamma_p}-\nu_1\cos\gamma_p\right)\cos\gamma_p\zeta-\nu_1\zeta\sin\gamma_p\sin\gamma_p\zeta\right]I_1\left(\frac{\gamma_p}{h}r\right) \tag{3}$$

$$2\mu W=-\left(\frac{1}{1+\nu}+\frac{a}{1-\nu}\right)h\zeta+\mathrm{Re}\sum_{p=1}^{\infty}D_p\left[\left(\frac{\nu_1+1}{\gamma_p}\sin\gamma_p+\nu_1\cos\gamma_p\right)\sin\gamma_p\zeta-\nu_1\zeta\sin\gamma_p\cos\gamma_p\zeta\right]I_0\left(\frac{\gamma_p}{h}r\right)$$

$$(\sin 2\gamma_p+2\gamma_p=0,\quad \mathrm{Re}\,\gamma_p>0,\quad \mathrm{Im}\,\gamma_p>0)$$

where the coefficients a and D_p are determined only by condition (2), since Eqs.(1) are identically satisfied when displacements of the cylinder points that obey formulas (3).

The substitution of the last formulas into the boundary condition (2) yields a functional equation in the variable ζ, whose algebraization is achieved by expansion in Fourier series

$$1,\ \cos\delta_m\zeta,\quad \sin\delta_m\zeta;\quad \delta_m=m\pi,\quad m=1,2\ldots \tag{4}$$

obtaining, as the result, a linear system in the unknown coefficients

$$\mathrm{Re}\sum_{p=1}^{\infty}D_p\frac{\gamma_p\sin^2\gamma_p}{(\gamma_p^2-\delta_m^2)^2}\left[(\nu_1-1)\gamma_p^2-(\nu_1+1)\delta_m^2\right]I_1\left(\frac{\gamma_p}{h}\right)=0 \tag{5}$$

$$(\nu_1-1)^2\Sigma+\frac{\nu_1+1}{2h}\,\mathrm{Re}\sum_{p=1}^{\infty}D_p\frac{\delta_m^2\sin^2\gamma_p}{(\gamma_p^2-\delta_m^2)^2}\times\left[(3\nu_1+1)\gamma_p^2-(\nu_1+1)\delta_m^2\right]I_0\left(\frac{\gamma_p}{h}\right)=-1$$

$$a=-\frac{2\nu^2}{1+\nu}+\frac{4\nu^2}{1-2\nu}\Sigma,\quad \Sigma=\mathrm{Re}\sum_{p=1}^{\infty}D_p\frac{\sin^2\gamma_p}{\gamma_p}I_1\left(\frac{\gamma_p}{h}\right),\quad m=1,2\ldots$$

*Prikl.Matem.Mekhan.,44,No.5,957-960,1980

The usual method of deriving an approximate solution of the problem formulated above is to curtail the resolving system (5). The upper solid line in Fig.1 shows the dependence of the discrepancy in boundary conditions

$$R = \max_{|\zeta|\leqslant 1} \{ |u_r(1,\zeta)|, |w(1,\zeta)| \} \tag{6}$$

on the order of the curtailed system K. All numerical data relate to the case of $\nu_1 = h = 2$ with r_0 and 2μ as the units of length and stress, respectively. The very slow convergence of the computational process depends in this case of the stress field singularity in the neighborhood of the boundary condition separation line /4,5/, since for any K the approximate solution is continuous throughout the region occupied by the elastic cylinder.

It is interesting to note that the sequence of approximate solutions qualitatively reflects the pattern of stress behavior near the above line. Curves of σ_r on the cylindrical surface computed with various accuracies appear in Fig.2. They represent a typical picture of the approximation of the discontinuous function by a sequence of continuous functions.

To improve the method convergence one can separate the principal part of the solution which determines the asymptotic behavior of coefficients D_p. Using the device proposed in /5/, we represent the coefficients with high subscripts as follows:

$$D_p \sim D\gamma_p^{\alpha} / I_0(\gamma_p / h) \tag{7}$$

To determine the asymptotics exponent α we obtained an equation using a computer. We fix some integer N, substitute into system (5) the expression (7) for coefficients with high exponent, and retain the first $2N+1$ equations. Then varying α we determine at each step the unknown $\alpha, D_p\ (1 \leqslant p \leqslant N), D$ and the discrepancy R. Curves of function $R = R(\alpha, N)$ are shown in Fig.1 by dash lines.

Numerical analysis had shown that the optimal value of α depends only on the Poisson coefficient. Comparing the latter with the order of stress singularities at the cylinder edge /4,5/

$$(3 - 4\nu)\sin^2(q\pi / 2) + q^2 = 4(1-\nu)^2, \quad 0 < q < 1 \tag{8}$$

we establish the important formula

$$\alpha = -q - 1 \tag{9}$$

Hence the exponent in formula (7) satisfies the equation obtained by the substitution of (9) into (8), and the coefficient D is determined together with $D_p\ (p \leqslant N)$ using the resolving system (5).

The proposed method makes possible a substantial increase of the convergence rate of approximate solutions (the lower solid curve in Fig.1). To obtain reasonably reliable results

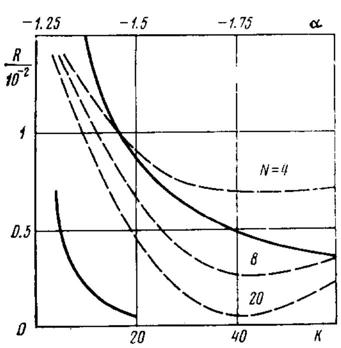

Fig.1

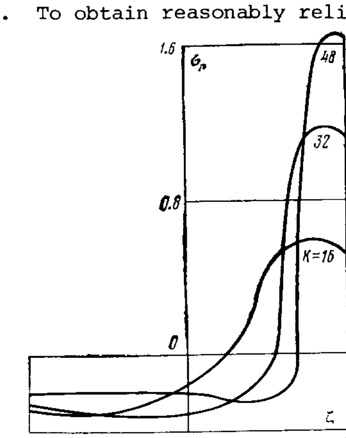

Fig.2

of numerical computations it is also, necessary to evaluate the convergence rate of series (3) for $r = 1$ (at inner points convergence is better).

For this we calculate the residual sum of the above series, retaining in the transforms quantities of higher order with respect to p

$$G(n) = (u_r + iw)_{p\geqslant n} \sim \nu_1 2^{\zeta}(1-\zeta) D \sum_{p=n}^{\infty} \delta_p^{\alpha+(1+\zeta)/2} \exp\left[i\pi \frac{(4p-1)(1-\zeta)+6}{4}\right], \quad n \gg N, \quad 0 \leqslant \zeta \leqslant 1$$

The coefficients of this trigonometric series monotonically decrease, hence they are convergent. Using the formula of adding trigonometric functions /6/

$$\sum_{j=1}^{m} \cos(b+jt) = \frac{\sin[b+(m+0.5)t] - \sin[b+0.5t]}{2\sin(t/2)}.$$

we obtain the required estimate

$$G(n) < \frac{\nu_1 2^{\zeta}(1-\zeta)}{\sin[\pi(1-\zeta)/2]} \, |D| \, \delta_n^{\alpha+(1+\zeta)/2}$$

The convergence rate of the computation process is important in investigations of multiply connected models because of the sharp increase in the expenditure of the computer operational memory with increasing approximation number. Extension of the described method to multiply connected regions is obvious.

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Translated by J.J.D.